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General stability conditions for a multi-layer model

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Sufficient conditions for the stability of steady solutions of a multi-layer model are found. The basic flow may be either parallel, axisymmetric or non-parallel. The lower boundary of the model may be either rigid, including the possibility of topography, or soft. The latter, 'reduced gravity', case represents an ideal situation in which the active layers are on top of an infinitely deep, motionless one.

Two conditions are sufficient to assure the stability of the basic flow. It is conjectured that unstable flows for which only the first or second condition is violated decay through Rossby-like or Poincaré-like growing perturbations, respectively.

In order to understand the meaning of both conditions, assume that a quite general $O(a)$ 'wave' is superimposed on the basic flow: an $O(a^2)$ energy integral, $\delta^2 E$ can be calculated. This wave energy is neither conserved, because the wave might exchange energy with the $O(a^2)$ varying part of the the 'mean flow', nor positive definite, because the perturbation might lower the total energy by increasing the speed where it decreases the thickness, and *vice versa*. Now, the first condition determines that $\delta^2 E$ has an upper bound, and the second one implies that $\delta^2 E$ is positive definite; hence the stability of the basic solution. In the particular case of two-dimensional divergenceless flow, as well as for quasi-geostropic models, $\delta^2 E$ is *a priori* positive definite, and therefore the first condition suffices to guarantee the stability of the more basic solution. The conditions found here are indeed valid for more general perturbations, e.g. they prevent inertial (or 'symmetric') instability, a phenomenon for which there is no distinction between wave and the varying part of the mean flow.

1. Introduction

One of the classical moves in the game of studying hydrodynamic stability consists in the search for conditions applicable to particular systems and flow patterns. Usually those conditions are ***sufficient*** for stability or, equivalently, ***necessary*** for instability (changing the logical sign of the statement). For any set of conditions – and the method used to obtain them – one can ask two questions. First, how powerful are they, in the sense of how close they are to being ***necessary and sufficient***. Second, what does the fulfilment, or violation, of a certain condition say about the physical properties of the basic flow and/or growing perturbations.

I shall address these questions here, for the particular case of multi-layer primitive equation models (figure 1). Their vertical resolution is classified by the parameter N, which may be integer or semi-integer: a value of $N = n+\frac{1}{2}$ layers, with integer n, actually means a model with $n+1$ layers in which the deepest one is at rest; it is, then, a system with n dynamic layers and a 'soft' lower boundary. On the other

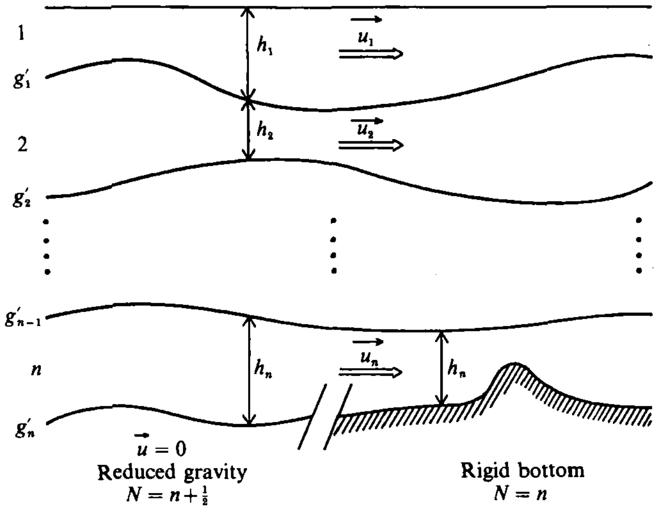

FIGURE 1. Structure of the N-layer model. The left part shows the 'reduced gravity' case (semi-integer N), and at the right is depicted the rigid bottom case (integer N). The parameters g'_j are the buoyancy jumps across the interfaces.

hand, in a regular N-layer model, i.e. with integer N, the deepest layer has a rigid bottom, which may include topography.

To set the context, consider a hydrostatic system with a single layer and reduced gravity g', i.e. a $1\frac{1}{2}$-layer model. If there exists any value of a constant α such that

$$N = 1\tfrac{1}{2}: \qquad (U(y)-\alpha)\frac{\mathrm{d}Q}{\mathrm{d}y} < 0 \tag{1a}$$

and

$$N = 1\tfrac{1}{2}: \qquad g'H(y) > (U(y)-\alpha)^2 \tag{1b}$$

for all y then the zonal flow $[u, v, h] = [U(y), 0, H(y)]$ is stable (Ripa 1983). (The notation is standard, for example Q, equal to $(f-\mathrm{d}U/\mathrm{d}y)/H$, denotes potential vorticity. The basic flow is in geostrophic balance: $fU+g'\,\mathrm{d}H/\mathrm{d}y = 0$.) In a model with two layers (Ripa 1987*b*, 1989), the second stability condition becomes (omitting the coordinate y for simplicity)

$$N = 2: \qquad g' > \frac{(U_1-\alpha)^2}{H_1}+\frac{(U_2-\alpha)^2}{H_2} \tag{1c}$$

if the system has a rigid bottom, which might include topography, or to

$$N = 2\tfrac{1}{2}: \qquad g_2'^2 < \left[g'_1+g'_2-\frac{(U_1-\alpha)^2}{H_1}\right]\left[g'_2-\frac{(U_2-\alpha)^2}{H_2}\right] \tag{1d}$$

in the 'reduced gravity' case (the lower boundary is the interface with a third, motionless layer; g'_1 and g'_2 denote the buoyancy jumps across both interfaces). The first stability condition is simply replaced, in either case, by

$$N = 2 \text{ or } 2\tfrac{1}{2}: \qquad (U_j-\alpha)\frac{\mathrm{d}Q_j}{\mathrm{d}y} < 0 \quad (j = 1, 2). \tag{2a,b}$$

These are sufficient stability conditions; necessary instability conditions are, therefore, that for any value of α (1) and/or (2) must be violated.

Two peculiarities of the stability conditions call the attention. Firstly, an arbitrary constant α may be subtracted from the fields $U_j(y)$: it is the variation of U that really matters. Thus uniform flow, $U =$ constant, is stable to finite-amplitude perturbations; for example there is no 'baroclinic instability' in a $1\frac{1}{2}$-layer model. Secondly, there are *two* different stability conditions, or two sets of them. This is in contrast to the cases of strictly two-dimensional horizontal flow ($N = 1$) or the *quasi-geostropic* models, for both of which there is only *one* stability condition.

The presence of the arbitrary constant α is not due to Galilean invariance, as is sometimes erroneously assumed, which is indeed a symmetry lost in the presence of Coriolis effects. Rather, it follows from zonal homogeneity of the basic flow under consideration, the evolution equations and boundary conditions (for example, a β-plane or sphere, with topography a function only of latitude, and with coasts along parallels). This property points towards a link between stability conditions and the symmetries of the problem: this relationship is clearly established when the former are obtained using the integrals of motion of the system, which in turn are related to its symmetries by Noether's theorem; this is the basis of Arnol'd's (1965, 1966) method (described in, for example, McIntyre & Shepherd 1987).

On the other hand, the second condition is linked to the importance of horizontal divergence. Violation of this condition by an unstable flow is associated with the possibility of perturbations with negative or vanishing energy (see for example Marinone & Ripa 1984; Hayashi & Young 1987; Ripa 1990). It is harder to satisfy ($1d$) than ($1c$), which, in turn, is more demanding than ($1b$). Indeed, Paldor & Killworth (1987) find that a finite bottom layer ($N = 2$), instead of an infinitely deep one ($N = 1\frac{1}{2}$), favours instability. More generally, the second condition is more easily violated in systems with richer vertical structure (larger N). It is therefore interesting to question what happens in the case of continuous stratification, which can be taken as the limit of an N-layer system, when $N \to \infty$.

Arnol'd's method is spelled out in §3 for the particular case of the multi-layer 'primitive equations' model, whose evolution equations and conservation laws are presented in §2. The stability conditions are presented in §4, the difficulties encountered when trying to generalize them from a finite number of layers to the continuum case are discussed in §5, and a conjecture on the structure of growing perturbations is presented in §6. The last section is reserved for a general discussion, and some mathematical details are presented in an Appendix.

2. The multi-layer model

The N-layer model of figure 1 is completely specified by the total volume of each layer (or the mean thickness, in the case of an unbounded horizontal domain) and the buoyancy jumps across the interfaces, $g'_j (g'_0 \equiv g)$. I am working with the (hydrostatic) primitive equations and making use of the Boussinesq approximation; one can easily manage without it, but that only complicates the notation without much gain in physical insight. The dynamic variables are the thickness and horizontal current in each layer, $\boldsymbol{h} \equiv \{h_j\}$ and $\boldsymbol{\vec{u}} \equiv \{\vec{u}_j\}$, the latter being independent of the vertical coordinate (the bold-face type denotes an n-vector in the space of layer variables, whereas the arrow indicates the position 2-vector in the horizontal plane). Independent variables are time t, the horizontal position $[x, y]$ and the layer label j.

Integral of motion	Symbol	Density, in each layer	Comment
Energy	E	$\frac{1}{2}h(u^2+v^2+p)$	$p \to p-p_n$ for rigid top & bottom
Casimir	C	$hF(q)$ where $q=\dfrac{f+\partial_x v-\partial_y u}{h}$	$F(\ldots)$: arbitrary potential vorticity
Momenta:			
Zonal	M	$h(u-f_0 y-\frac{1}{2}\beta y^2)$	β-plane
Linear	$\vec{M}$	$h(u-fy, v+fx)$	f-plane
Angular	A	$h[xv-yu+\frac{1}{2}f(x^2+y^2)]$	f-plane

TABLE 1. Integrals of motion for system (3)

The equations of motion are (omitting the layer subscript for simplicity)

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv+\frac{\partial p}{\partial x}&=0,\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu+\frac{\partial p}{\partial y}&=0,\\ \frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y}&=0.\end{aligned}\right\} \qquad (3)$$

The kinematic pressures and layer thicknesses are related, through the vertical displacements ζ_j, by the hydrostatic and incompressibility conditions

$$p_{j+1}-p_j = g'_j\zeta_j, \quad \partial h_j = \partial(\zeta_{j-1}-\zeta_j), \qquad (4a, b)$$

where ∂ denotes any horizontal or time derivative. These relations are closed by suitable bottom and top boundary conditions. First is the requirement of either 'reduced gravity' dynamics: $N = n+\frac{1}{2} \Rightarrow \nabla p_{n+1} \equiv 0$, or rigid bottom: $N = n \Rightarrow \partial\zeta_n/\partial t \equiv 0$ (including the possibility of topography). Secondly, $p_0 \equiv 0$ (no atmospheric pressure forcing); this could be replaced by the 'rigid lid' approximation which corresponds to the limit

$$g \to \infty, \quad \zeta_0 \to 0 \quad \text{but} \quad g\zeta_0\,(=p_1) = \text{finite}. \qquad (5)$$

Thus p_j is a linear function of the h_k, except in the case when both boundaries are rigid, whence p_j-p_n, say, is a linear function of the h_k (see Appendix).

The integrals of motion for this system are given in table 1. Energy and momentum conservation are related to time and space symmetries of the evolution equations and boundary conditions: if there are coasts or topography that lack a particular symmetry, then the corresponding momentum is not conserved. The family of 'Casimir' integrals of motion, on the other hand, is associated with relabelling invariance, in a Lagrangian description of the flow (Ripa 1981; Salmon 1982), or to the property of Eulerian equations of being non-canonical in the Hamiltonian formalism (Shepherd 1990). (Other integrals of motion, like the volume in each layer – or the average thickness, in the case of an unbounded domain – and the circulation around rigid walls, are but special cases of Casimirs.) The proof that all these are indeed constants of motion is rather trivial, with the exception of the potential energy, for which equation (A 10) in the Appendix is useful.

The law of energy conservation then takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{n}\iint \mathrm{d}x\,\mathrm{d}y\,\tfrac{1}{2}h_j(\vec{u}_j^2+p_j)=0; \tag{6}$$

using (4) and (A 1), the potential energy term can be rewritten as (recall that p_j-p_n should be used, instead of p_j, in the case of rigid top and bottom)

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint \mathrm{d}x\,\mathrm{d}y\sum_{j=1}^{n}h_j\,p_j=\frac{\mathrm{d}}{\mathrm{d}t}\iint \mathrm{d}x\,\mathrm{d}y\sum_{j=0}^{[N+\frac{1}{2}]}g_j'\,\zeta_j^2.$$

(In deriving this equation, terms linear in p_j or p_j-p_n are generated, whose time derivative vanishes, by virtue of (A 5) or (A 9) and the mass conservation law (3c)). The upper limit of the sum on the right-hand side equals n in the rigid-bottom case and $n+1$ in the reduced-gravity one; notice that if the rigid-lid approximation (5) is made, then the first term, $g\zeta_0^2$, vanishes.

Equation (6) is written in terms of the total fields; splitting these, instead, as the sum of a (steady) basic solution plus the departure from it (the perturbation),

$$\vec{\boldsymbol{u}}=\vec{\boldsymbol{U}}(\vec{x})+\delta\vec{\boldsymbol{u}}(\vec{x},t),\quad \boldsymbol{h}=\boldsymbol{H}(\vec{x})+\delta\boldsymbol{h}(\vec{x},t), \tag{7}$$

the energy density in each layer is expanded as (omitting the subscript for simplicity)

$$\tfrac{1}{2}h(\vec{u}^2+p)=\tfrac{1}{2}H(\vec{U}^2+P) \tag{8a}$$

$$+H\vec{U}\cdot\delta\vec{x}+(\tfrac{1}{2}\vec{U}^2+P)\,\delta h \tag{8b}$$

$$+\tfrac{1}{2}H\,\delta\vec{u}^2+\vec{U}\cdot\delta\vec{x}\,\delta h+\tfrac{1}{2}\delta h\,\delta p \tag{8c}$$

$$+\tfrac{1}{2}\delta h\,\delta\vec{u}^2. \tag{8d}$$

In the derivation of (8b), $\sum H_j\,\delta p_j=\sum P_j\,\delta h_j$ was used, which can be derived from (4), as is equation (A 10) in the Appendix.

It is clear that the first line on the right-hand side, (8a), does not contribute to (6) because it is time independent, by construction. The integral of (8b) does not necessarily vanish, even though it is linear in the perturbation, because even if the latter were mainly a 'wave' with zero mean, it will also include the varying part of the 'mean flow'. Therefore, the integral of (8c) is not conserved.

As an example – just as an example – consider the case of a zonal basic flow, $[U(y), 0, \boldsymbol{H}(y)]$, and split the perturbation in (7) in the form

$$\left.\begin{aligned}\delta\boldsymbol{h}&=\boldsymbol{h}_{\mathrm{m}}(y,t)+\boldsymbol{h}_{\mathrm{w}}(x,y,t),\\ \delta\vec{\boldsymbol{u}}&=\vec{\boldsymbol{u}}_{\mathrm{m}}(y,t)+\vec{\boldsymbol{u}}_{\mathrm{w}}(x,y,t),\end{aligned}\right\} \tag{9}$$

where the subscripts m and w denote zonal average and deviation from it, respectively. Let $\{\vec{\boldsymbol{u}}_{\mathrm{w}},\boldsymbol{h}_{\mathrm{w}}\}$ be, to lowest order, an $O(a)$ wave; then, $\{\vec{\boldsymbol{u}}_{\mathrm{m}},\boldsymbol{h}_{\mathrm{m}}\}$ represents the $O(a^2)$ variability of the mean flow. Each one will be the main contributor to the integral of (8b) and (8c), respectively; that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{n}\iint \mathrm{d}x\,\mathrm{d}y[\underbrace{\{H\vec{U}\cdot\vec{u}_{\mathrm{m}}+(\tfrac{1}{2}\vec{U}^2+P)\,h_{\mathrm{m}}\}}_{\text{Mean flow energy}}+\underbrace{\{\tfrac{1}{2}H\vec{u}_{\mathrm{w}}^2+\vec{U}\cdot\vec{u}_{\mathrm{w}}\,h_{\mathrm{w}}+\tfrac{1}{2}h_{\mathrm{w}}\,p_{\mathrm{w}}\}}_{\text{Wave energy}}]=0$$

(up to $O(a^3)$) represents no more than the energy exchange between the mean flow and the wave.

3. Arnol'd's method

The constants of motion in Table 1 are used with the procedure devised by Arnol'd (1965, 1966) in order to find sufficient stability conditions. The method consists of constructing a certain integral of motion S and finding under what conditions it has an extremum for a particular basic flow; those conditions guarantee the stability of such an equilibrium solution. More precisely, let $\{\vec{U}, \boldsymbol{H}\}$ be an exact (nonlinear) solution of the equations of motion, then, if $S[\vec{u}, \boldsymbol{h}] =$ constant, for any initial condition, and

$$\Delta S \equiv S[\vec{U}+\delta\vec{u}, \boldsymbol{H}+\delta\boldsymbol{h}]-S[\vec{U}, \boldsymbol{H}] > 0$$

for all finite perturbations $\{\delta\vec{u}, \delta\boldsymbol{h}\}$ that are different from zero and sufficiently small, the basic flow $\{\vec{U}, \boldsymbol{H}\}$ is stable. Notice that $S[\vec{U}, \boldsymbol{H}]$ is constant because the basic flow is assumed steady. (If a certain S were maximum at $\{\vec{U}, \boldsymbol{H}\}$, instead of minimum, then by changing its sign it is cast in the form $\Delta S > 0$).

Now, let me expand the total variation ΔS into that part which is linear in the perturbation $\{\delta\vec{u}, \delta\boldsymbol{h}\}$, δS, that part which is quadratic, $\delta^2 S$, etc. (Namely, $\Delta S = \delta S+\delta^2 S+\delta^3 S+\ldots$, where $\delta^n S = O(\delta\vec{u}, \delta\boldsymbol{h})^n$.) Clearly, $\Delta S > 0$ requires

$$\delta S \equiv 0 \quad \text{and} \quad \delta^2 S > 0, \quad \forall\{\delta\vec{u}, \delta\boldsymbol{h}\} \neq \{0,0\}, \tag{10a, b}$$

because otherwise if δS were, say, positive for a certain $\{\delta\vec{u}, \delta\boldsymbol{h}\}$, then it would be negative for $\{-\delta\vec{u}, -\delta\boldsymbol{h}\}$; positive definiteness of $\delta^2 S$ follows from requiring $\Delta S > 0$ for an infinitesimal perturbation. Instead of $\Delta S > 0$ I shall use the weaker condition (10b).

The functional S cannot be just the regular energy, since $\mathrm{d}E/\mathrm{d}t = 0$, equation (6), does not bound the growth of the perturbation, which is what is meant here by stability of the basic flow, because $\delta E \neq 0$. In the simplest case, S is the *pseudoenergy*, which is defined as that combination of the regular energy and a Casimir, $S = E+C$, which makes ΔS, to lowest order, quadratic in the perturbation, i.e. such that (10a) is satisfied.

The partition (7) gives, for the potential vorticity in each layer (omitting the layer subscript for simplicity), $q = Q+\delta\xi/h$, exactly, with

$$\delta\xi \equiv \partial_x \delta v-\partial_y \delta u-Q\,\delta h, \tag{11}$$

and, therefore,

$$q = Q+\frac{\delta\xi}{H}-\frac{\delta\xi\,\delta h}{H^2}+O(\delta\xi^3, \delta h\,\delta\xi^2).$$

Using this in the Casimir conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{n}\iint \mathrm{d}x\,\mathrm{d}y h_j F_j(q_j) = 0, \tag{12}$$

gives, in each layer, the expansion

$$\begin{aligned} hF(q) = {} & HF(Q) && (13a)\\ & +F(Q)\,\delta h+F'(Q)\,\delta\xi && (13b)\\ & +\tfrac{1}{2}[F''(Q)/H]\,\delta\xi^2 && (13c)\\ & +O(\delta\xi^3, \delta h\,\delta\xi^2). && (13d) \end{aligned}$$

Once again, the right-hand side of (13a) does not contribute to (12) because it is time invariant, by construction. The trick is to choose $F(q)$ so that, adding (6) and (12), the integral of (8b) and (13b) cancel out, $\delta E+\delta C \equiv 0$, and one is left with an integral of motion (the pseudoenergy) which is quadratic, to lowest order, in the perturbation.

It might seem that such a function $F(q)$ could only be found in exceptional cases, because it is determined by two simultaneous equations. However, that is indeed not the case, because one of those equations is but the derivative of the other: the only condition needed in order to find $F(q)$, and therefore construct the pseudoenergy, is $\partial_t\{\vec{U}, H\} \equiv 0$ (i.e. for the basic flow to have the symmetry associated with energy conservation): the balances of a steady solution yield $F(q) = q\Psi(q)-B(q)$, where $\Psi(Q)$ is the transport function and $B(Q)$ ($\equiv \frac{1}{2}\vec{U}^2+P$) is the Bernoulli function in the basic state.

Pseudoenergy conservation then reads

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Delta E+\Delta C) = 0,$$

where

$$\Delta E \equiv \sum_{j=1}^{n} \iint \mathrm{d}x\,\mathrm{d}y\{\tfrac{1}{2}(H+\delta h)\,\delta\vec{u}^2+\vec{U}\cdot\delta\vec{u}\,\delta h+\tfrac{1}{2}\delta h\,\delta p\}, \tag{14}$$

and

$$\Delta C \equiv \sum_{j=1}^{n} \iint \mathrm{d}x\,\mathrm{d}y[h(F(q)-F(Q))-F'(Q)\,\delta\xi]$$

$$= \sum_{j=1}^{n} \iint \mathrm{d}x\,\mathrm{d}y\frac{F''(Q')}{2H}\delta\xi^2+O(\delta\xi^3, \delta h\,\delta\xi^2). \tag{15}$$

If one can prove that ΔC is positive definite for small enough perturbations (i.e. $\delta^2 C$ positive definite; that is the purpose of the first condition) and equally so for ΔE (similarly for the second condition), then

$$\boxed{\begin{array}{c} \delta^2E(0)+\delta^2C(0) > \underset{\substack{\Uparrow \\ 1^{\text{st}}}}{\delta^2E(t)} > \underset{\substack{\Uparrow \\ 2^{\text{nd}}}}{0} \end{array}} \tag{16}$$

and

$$\boxed{\begin{array}{c} \delta^2E(0)+\delta^2C(0) > \underset{\substack{\Uparrow \\ 2^{\text{nd}}}}{\delta^2C(t)} > \underset{\substack{\Uparrow \\ 1^{\text{st}}}}{0} \end{array}} \tag{17}$$

'Small enough perturbation' means such that the quadratic terms, δ^2E and δ^2C, dominate in the law of pseudoenergy conservation $\mathrm{d}(\Delta E+\Delta C)/\mathrm{d}t = 0$.

Consequently, the first and second conditions imply that the wave energy and wave Casimir are bounded, at any time, by zero from below and by the value of the initial pseudoenergy from above, as indicated by the arrows. Equation (16) shows that the growth of the perturbation is bounded in an energy metric defined by the integral of (8c); hence the stability of the basic flow. Similarly, (17) bounds the growth of the perturbation potential vorticity, in a metric defined by the integral of (13c).

I shall now discuss the breadth of these results. First, the perturbation can be somewhat more general than what is exemplified in (9), for example the conditions also prevent inertial (or 'symmetric') instability; that is the reason for using the total perturbation, $\{\delta\vec{\boldsymbol{u}}, \delta\boldsymbol{h}\}$ and not just $\{\vec{\boldsymbol{u}}_{\mathrm{w}}, \boldsymbol{h}_{\mathrm{w}}\}$.

Secondly, the stability conditions guarantee 'formal' stability, which in turn implies the weaker properties of 'linear' and 'spectral' stability, in the notation of Holm *et al.* (1985) (see also McIntyre & Shepherd 1987). In order to prove truly 'nonlinear' or 'normed' stability, i.e. in the Lyapunov sense, it is necessary to make convexity estimates, which guarantee that the perturbation is completely bounded by a factor of its original size, in a suitable measure. Normed stability has only been proved, to my knowledge, for the $N = 1$ case and the quasi-geostropic models, systems for which ΔE is exactly quadratic, unlike the problem considered here; this is an unfortunate limitation of ageostropic three-dimensional dynamics, given the power of normed stability (e.g. see Shepherd 1988*b*).

Thirdly, I have mentioned the possibility of pseudoenergy being a maximum at the basic state ($S = E+C$), and not a minimum ($S = -E-C$). The latter, so called Arnol'd's (1966) second theorem, has only been proved, once again, for the $N = 1$ case and the quasi-geostrophic models, the reason being that those systems have only one independent dynamical field, the vorticity, and therefore $\delta^2 E$ and $\delta^2 C$ can be related, for an arbitrary perturbation. This is not the case here: δq does not determine $\{\delta\vec{\boldsymbol{u}}, \delta h\}$.

Finally, three types of velocity field $[U_j, V_j]$ for the basic flow can be considered: *parallel*, *axisymmetric* and *non-parallel*.

The parallel case,

$$U_j = U_j(y), \quad V_j \equiv 0, \tag{18}$$

corresponds to a system homogeneous in x (e.g. a β-plane or sphere, with topography a function only of latitude, and with coasts along parallels).

The axisymmetric flow,

$$[U_j, V_j] = \Omega_j(r)\,[-y, x], \tag{19}$$

is a possible solution in a system with horizontal isotropy (e.g. an f-plane in a circular or unbounded domain, and with any topography, a function only of $r\,(\equiv (x^2+y^2)^{\frac{1}{2}})$.

The non-parallel case,

$$H_j[U_j, V_j] = [-\partial_y, \partial_x]\,\Psi_j(x, y), \tag{20}$$

corresponds to a system that is neither homogeneous nor isotropic (e.g. a β-plane with meridional coasts or an isolated topographic feature).

It may seem unnecessary to single out the symmetric cases, (18) and (19), since those flows are certainly particular limits of the non-parallel one, (20). However, this is not so, for two important reasons.

First, in the symmetric cases we can get better stability conditions by including a term proportional to the corresponding momentum in the definition of the functional S, namely

parallel $\qquad S = E - \alpha M + C, \qquad (21a)$

axisymmetric: $\qquad S = E - \omega A + C, \qquad (21b)$

non-parallel: $\qquad S = E + C, \qquad (21c)$

where α and ω are arbitrary constants. Thus in the parallel (axisymmetric) case the Casimir C is chosen so that δC cancels out $\delta E - \alpha\delta M$ ($\delta E - \omega\delta A$), not just δE, as in the non-parallel case. Conditions derived requiring $\delta^2 S > 0$ from (21*a*) or (21*b*) are more powerful than that derived from (21*c*) on account of the arbitrary constant α or ω.

If the basic flow is *not* symmetric, even though some total momentum may be conserved, say M, it is not possible to choose $F(q)$ so that δC cancels out δM: (21*a*) and (21*b*) require $\partial_x\{\vec{U}, H\} \equiv 0$ and $(y\partial_x - x\partial_y)\{\vec{U}, H\} \equiv 0$, respectively (i.e. for the basic flow to have the symmetry associated with each momentum conservation).

It can be noted that the expression for $F(q)$ depends upon the particular form of the basic flow: in the non-parallel case, (21*c*), it is $F(q) = q\Psi(q) - B(q)$, as explained above; in both symmetric cases, (21*a*) or (21*b*), to this expression for $F(q)$, $\alpha\int Y(q)\,\mathrm{d}q$ or $-\frac{1}{2}\omega\int R^2(q)\,\mathrm{d}q$, must be added where $Y(Q)$ and $R(Q)$ are the inverse of $Q(y)$ and $Q(r)$, respectively.

Second, if the system (model equations and boundary conditions) has a certain symmetry, then there are no non-parallel solutions of $\delta^2(E+C) > 0$ (Andrews 1984); Arnol'd stable solutions must have some symmetry, in which case use of the corresponding momentum, as in (21*a*) and (21*b*) gives a better stability condition. Briefly, the proof of that statement is as follows. Assume that the system is x-homogeneous: this implies that if $\phi(x, y, t)$ is a solution, where $\phi \equiv \{\vec{u}, h\}$, then $\phi(x+\delta x, y, t)$ must also be a solution for any δx. Since the integrals that define both E and C are also independent of a translation in x, then $\delta^2(E+C) > 0$ cannot exist for the particular perturbation $\delta\phi \equiv \Phi(x+\delta x, y) - \Phi(x, y)$, where Φ is any steady nonlinear solution: consequently, $\partial_x \Phi$ must vanish.

Now, assume that a certain system has *two* symmetries, say, x-homogeneity and horizontal isotropy. An Arnol'd stable flow must be either parallel, in which case it will be a solution of $\delta^2(E - \alpha M + C) > 0$ with $\alpha \neq 0$, or axisymmetric, in which case it will be a solution of $\delta^2(E - \omega A + C) > 0$ with $\omega \neq 0$. Notice (table 1) that the integral $E - \alpha M + C$ is not invariant under rotation, neither is the integral $E - \omega A + C$ invariant under translation, and therefore there is no contradiction with Andrew's theorem; rather, this is a corollary of it. As an example, a $1\frac{1}{2}$-layer circular vortex in anticyclonic solid-body rotation is proved to be stable, by showing that $\delta^2(E - \omega A + C) > 0$ with ω equal to its angular velocity (Ripa 1987*a*), but $\delta^2(E+C)$ *is not* sign definite.

4. Sufficient stability conditions

The *first stability condition* is the one that assures that $\delta^2 C$ is positive definite, and therefore that $\delta^2 E$ is bounded from above, see (16). This condition is $F''(Q) > 0$ everywhere, see (15), and is presented in the middle column of the upper part of table 2; it must be satisfied in every layer ($j = 1, \ldots, n$) and in all horizontal positions.

The *second stability condition* is the one that assures that $\delta^2 E$ is positive definite, and therefore that $\delta^2 C$ is bounded from above; see (17). Unlike in the quasi-geostrophic models and the $N = 1$ case, for which $h_j =$ constant is used in the kinetic energy integral, $\delta^2 E$ need not be positive definite: There might exist perturbations with 'negative energy', i.e. such that the perturbed state has less energy than the basic one, because the perturbation manages to decrease (increase) the total speed where it thickens (thins) the layer. Rewriting the leading order of the integrand of (14) as

$$\tfrac{1}{2}H\,\delta\vec{u}^2 + \vec{U}\cdot\delta\vec{u}\,\delta h + \tfrac{1}{2}\delta h\,\delta p = \tfrac{1}{2}H\left(\delta\vec{u} + \frac{\vec{U}}{H}\delta h\right)^2 + \frac{1}{2}\left\{\delta h\,\delta p - \frac{\vec{U}^2}{H}\delta h^2\right\},$$

Basic flow	$[U_j, V_j]$	First condition	Second condition
Parallel	$[U_j, 0]$	$\dfrac{U_j-\alpha}{\mathrm{d}Q_j/\mathrm{d}y} < 0$	$\dfrac{(U_j-\alpha)^2}{H_j} < \mu_j$
Axisymmetric	$\Omega_j(r)[-y, x]$	$\dfrac{\Omega_j-\omega}{\mathrm{d}Q_j/\mathrm{d}r} > 0$	$\dfrac{(\Omega_j-\omega)^2 r^2}{H_j} < \mu_j$
Non-parallel	$H_j^{-1}[-\partial_y, \partial_x]\,\Psi_j(x,y)$	$\dfrac{\mathrm{d}\Psi_j}{\mathrm{d}Q_j} > 0$	$\dfrac{U_j^2+V_j^2}{H_j} < \mu_j$

Auxiliary fields

	Parallel	Axisymmetric	Non-parallel
$\gamma_j \equiv$	$(U_j-\alpha)^2/H_j$	$(\Omega_j-\omega)^2 r^2/H_j$	$(U_j^2+V_j^2)/H_j$

$\mu_n \equiv g'_n \quad (N = n+\frac{1}{2})$ or $\mu_n \equiv \infty \quad (N = n)$

and

$$\mu_j = g'_j - \frac{\mu_{j+1}\gamma_{j+1}}{\mu_{j+1}-\gamma_{j+1}}, \quad j = n-1, \ldots, 1$$

TABLE 2. Stability conditions

it follows that $\delta^2 E$ – or $\delta^2(E-\alpha M)$ or $\delta^2(E-\omega A)$ in the parallel or axisymmetric cases, respectively – is positive definite if, and only if,

$$\sum_{j=1}^{n} (\delta h_j \delta p_j - \gamma_j \delta h_j^2) > 0 \quad \forall \delta \boldsymbol{h} \neq 0, \tag{22}$$

where the fields γ_j are defined as $(U_j-\alpha)^2/H_j$, $(\Omega_j-\omega)^2/H_j$ and $(U_j^2+V_j^2)/H_j$ in the parallel, axisymmetric and non-parallel cases, respectively.

Thus, in the $1\frac{1}{2}$-layer case, for which simply $\delta p = g'\,\delta h$, (22) is satisfied if and only if $g' > \gamma$, i.e. (1*b*) for parallel flow. Furthermore, in the 2-layer case, (22) requires $g' > \gamma_1 + g_2$, i.e. (1*c*) for parallel flow, because the rigid lid and rigid bottom boundary conditions require $\delta h_1 + \delta h_2 = 0$.

In the general case, it is better to make the rigid-lid approximation and rewrite (22) in terms of $\delta\zeta_j$ instead of δh_j. The first term, potential energy, is positive definite: from (4) it is $\delta p_{j+1} - \delta p_j = g'_j \delta\zeta_j$ and $\delta h_j = \delta\zeta_{j-1} - \delta\zeta_j$; rearranging sums (equivalent to a partial vertical integration) and using the appropriate top and bottom boundary condition, it follows that

$$\sum_{j=1}^{n} \delta h_j \delta p_j = \sum_{j=1}^{[N+\frac{1}{2}]} g'_j \delta\zeta_j^2;$$

the sea-surface elevation term in the right-hand side, $g\,\delta\zeta_0^2$, is excluded because of the rigid-lid approximation (5). (Recall that $p_j - p_n$ should be used, instead of p_j, if both top and bottom boundaries are rigid.) The condition of positive definiteness (22) is then found equivalent to requiring that all the eigenvalues of a tri-diagonal matrix be positive (see, for instance, Bloom 1979). This way, the second stability condition is the one presented in the right column of the upper part of table 2, with the auxiliary fields μ_j and γ_j defined in the bottom part of that table. (If the rigid-lid approximation were not used, then the only change to be made in the definition of the μ_j is to replace g'_1 by $g'_1 - \gamma_1^2/(g-\gamma_1)$: since γ_1 is at most $O(g'_1)$ this correction is insignificant for $N > 1$. In the context of this work, the validity of the rigid-lid approximation is based in neglecting the sea-surface contribution to both the total

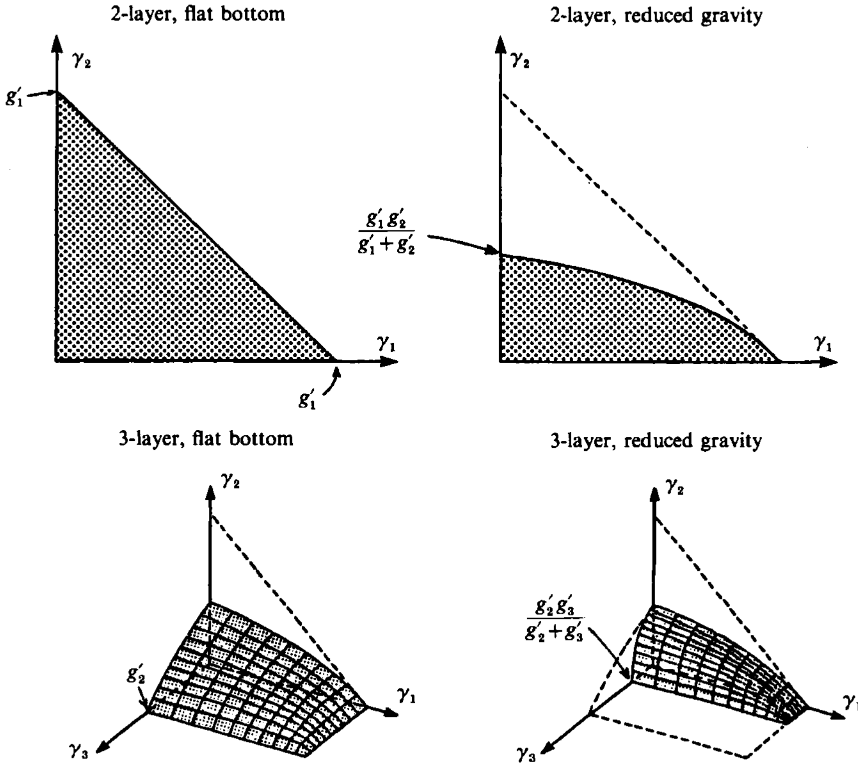

FIGURE 2. Region where the second stability condition (wave energy–momentum positive definite) is satisfied. The layer variables γ_j are defined as $(U_j-\alpha)^2/H_j$, $(\Omega_j-\omega)^2/H_j$ and $(U_j^2+V_j^2)/H_j$ in the parallel, axisymmetric and non-parallel cases, respectively.

N	Second condition
1	Always
$1\frac{1}{2}$	$g' > \gamma$
2	$g' > \gamma_1+\gamma_2$
$2\frac{1}{2}$	$g_2'^2 < (g'_1+g'_2-\gamma_1)(g'_2-\gamma_2)$
3	$(g'_2-\gamma_3)^2 < (g'_1+g'_2-\gamma_1-\gamma_3)(g'_2-\gamma_2-\gamma_3)$
$3\frac{1}{3}$	$\mu^2 < (g'_1-\gamma_1+\mu)(\mu-\gamma_2)$, with $\mu = g'_2-g'_3\gamma_3/(g'_3-\gamma_3)$

Type of basic flow:

	Parallel	Axisymmetric	Non-parallel
$\gamma_j \equiv$	$(U_j-\alpha)^2/H_j$	$(\Omega_j-\omega)^2/H_j$	$(U_j^2+V_j^2)/H_j$

Table 3. The second stability condition for systems where $N < 4$

potential energy and the divergence in the upper layer.) The second condition must be satisfied for all horizontal positions, and for $(j = 1, \ldots, n)$ in the reduced-gravity case or $(j = 1, \ldots, n-1)$ in the rigid-bottom one. This condition requires

$$\gamma_j < \mu_j \leqslant g'_j.$$

As a check, notice that one goes from $N = n+1$ to $N = n+\frac{1}{2}$ by making $H_{n+1} \to \infty$, which implies $\gamma_{n+1} \to 0$. Similarly, one formally goes from $N = n+\frac{1}{2}$ to $N = n$ by making $g'_n \to \infty$. I say 'formally' because the n-layer model might have topography, and therefore is not truly derivable from the $n+\frac{1}{2}$-layer one.

The second condition, for systems with few layers ($N < 4$) is spelled out in table 3 and presented in figure 2. It is clear in that figure that the second condition (wave energy momentum positive definite) is more restrictive on the shear of the basic flow, as the number of layers is increased.

The sufficient stability conditions presented in table 2 are the generalization of (1*a*) and (1*b*) for the multi-layer case. Necessary instability conditions are the following. *Non-parallel case*: Either the first or second condition, or both, must be violated somewhere. *Parallel case*: *Idem*, for *any* α. *Axisymmetric case*: *Idem*, for *any* ω.

5. From one to infinity: the ultraviolet problem

A multi-layer system may be seen as a particular discretization of a continuously stratified one in a mixed formulation: Eulerian in the horizontal and Lagrangian in the vertical. (Not to be confused with the fully Eulerian one: the pseudoenergy $\Delta E + \Delta C$, or even $\delta^2 E + \delta^2 C$, is the same in both formalisms, but that is not true for the wave energy $\delta^2 E$, which is positive definite in the latter and sign indefinite in the former (Ripa 1990).) More precisely, the vertical coordinate σ is taken as the reference depth of each isopycnal; the actual depth equals σ plus the vertical displacement ζ. The momentum and continuity equations are once again (3), whereas (4) is replaced by (e.g. Ripa 1981, 1990)

$$\frac{\partial p}{\partial \sigma} = -\mathscr{N}^2(\sigma)\,\zeta, \quad h \equiv 1 + \frac{\partial \zeta}{\partial \sigma}, \tag{23}$$

where $\mathscr{N}$ is the Brunt-Väisälä frequency ($\mathscr{N}^2(\sigma)$ is the vertical gradient of the reference buoyancy profile).

It is then interesting to question whether the multi-layer stability conditions can be generalized to the continuously stratified case. This is clearly possible with the first condition (middle column of table 2); that is not the case at all with the second condition (right column). Assume that successive approximations of a continually stratified system are built by N-layer models, with increasing N. As the layer thicknesses are diminished, say $H \rightarrow \epsilon H$, with $\epsilon < 1$, so are the buoyancy jumps, $g' \rightarrow \epsilon g'$, but the fields γ are increased, $\gamma \rightarrow \gamma/\epsilon$, and thus the second condition is harder to satisfy, by a factor of ϵ^{-2}. The inevitable conclusion is that there are no conditions on the basic flow that could assure that the wave energy or momentum of an arbitrary perturbation be positive definite: one has to resort to conditions that also involve properties of the disturbance.

For instance, Holm & Long (1989) found, in the continuum system,

$$\frac{\partial \Psi}{\partial Q} > 0, \quad \frac{\mathscr{N}^2}{m^2} > U^2 + V^2 \tag{24a, b}$$

for the stability of the non-parallel case, where m is a local vertical wavenumber of the perturbation; immediate applications to the parallel and axisymmetric cases are

$$\frac{U-\alpha}{\partial Q/\partial y} < 0, \quad \frac{\mathscr{N}^2}{m^2} > (U-\alpha)^2 \tag{25a, b}$$

for some α and

$$\frac{\Omega-\omega}{\partial Q/\partial r} > 0, \quad \frac{\mathscr{N}^2}{m^2} > (\Omega-\omega)^2 r^2 \tag{26a, b}$$

for some ω. Unlike the conditions in table 2, which only involve the basic flow, (24)–(26) also limit the vertical scale of the perturbations: short enough disturbances will violate it. In other words, for an *unstable* flow that satisfies (24*a*), (25*a*) for some α, or (26*a*) for some ω, whichever corresponds, the vertical scale of a growing perturbation must be small enough that (24*b*), (25*b*) or (26*b*) is violated. This 'ultraviolet' problem was first discussed by Blumen (1971), using normal modes in the pseudoenergy integral.

Therefore, (24)–(26) are conditions for *normal-mode* stability (a claim otherwise by Holm & Long (1989) not withstanding), since in a general time-dependent problem, there is no way to assure that m^2 will be bounded at all times, even if it were so initially, as clearly pointed out by Carnevale *et al.* (1988) and Shepherd (1988*a*). Notice that even in some linear cases the maximum of m^2 might grow with time, as in the case of a wave packet approaching a critical layer.

Conditions (24*b*)–(26*b*) are not directly related to the corresponding one in table 2, but, rather, with a requirement of *subcriticality*, which is more demanding, i.e. it is

$$\text{Subcriticality} \Rightarrow 2^{\text{nd}} \text{ condition},$$

but *not* vice versa. In order to see this in the context of the multi-layer model (finite N), assume that the perturbation were expanded, at each horizontal position, in terms of the vertical normal modes determined by the local density stratification of the basic state (e.g. see Ripa 1986): equation (22) is found to be satisfied, using the Cauchy–Schwartz inequality, if U^2+V^2, $(U\text{-}\alpha)^2$ or $(\Omega-\omega)^2r^2$ is smaller than the minimum of λ, where $\{\lambda\}$ is the spectrum of eigenvalues of the vertical modes. Now, that minimum of λ tends to zero as $N\rightarrow\infty$, hence the existence of the ultraviolet problem also for the subcriticality condition, as expected (i.e. violation of the second condition implies likewise for the subcriticality one, but not otherwise).

For $N = 1\frac{1}{2}$ (and the particular case of a barotropic basic flow) both formulations are equivalent (i.e. in (2*a*) it is $\lambda^2 \equiv g'H$) but in systems with more than one layer, that is no longer true, as it can be appreciated from figure 3. For instance, in the 2-layer analysis of hydrostatic Kelvin–Helmholtz instability (Ripa 1990), the subcriticality condition $(U_1-U_2)^2 \leqslant 4g'H_1H_2/(H_1+H_2)$, is clearly more restrictive than the second stability condition

$$(U_1-U_2)^2 \leqslant g'(H_1+H_2) \tag{27}$$

(here expressed for the optimum value of α), unless both layers happen to be equally deep.

The conditions derived from conservation laws might be too restrictive as the vertical resolution is improved: using the equation for a normal-mode perturbation to a shear flow in a vertical plane without rotation, the two-dimensional Boussinesq problem, both Miles (1961) and Howard (1961) found $Ri\,(\equiv \mathcal{N}^2/U_z^2) > \frac{1}{4}$ as stability condition (Ri is the Richardson number). This equation does not involve the scale of the perturbation, unlike (25*b*), but, unfortunately, it is obtained from an integral which is *not* the perturbation energy integral, or any other integral of motion. (The condition of Miles–Howard can be related to the energetics of particle interchange in the vertical though (Drazin & Reid 1981), but not to a field energy integral; the kinematics and dynamics of such exchange are ignored in that heuristic argument.)

Abarbanel *et al.* (1986) assumed a horizontal flow of the form $U(z)+\epsilon y^2$ in order to break the potential vorticity degeneracy, and found $U/U_{yy} > 0$ and $Ra\,(\equiv 2\mathcal{N}^2/[(U^2)_{\rho\rho}\,(\rho_z)^2]) > 1$ as nonlinear stability conditions, to lowest order in ϵ. Now,

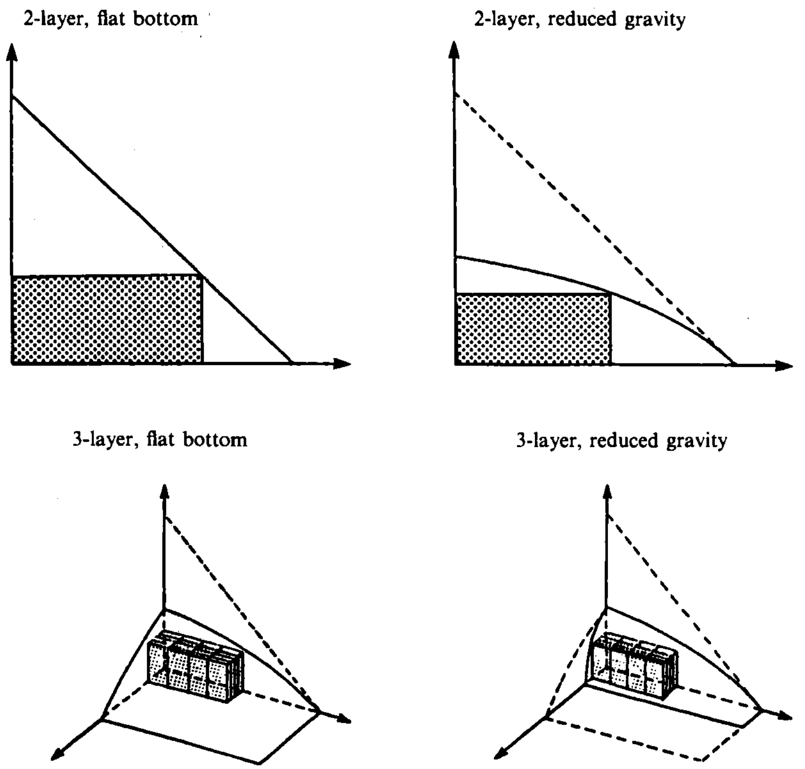

FIGURE 3. Region where the subcriticality condition is satisfied; this is much more restrictive than that of figure 2, with the exceptions of of the $N = 1\frac{1}{2}$ model and the particular case of a barotropic basic flow.

$Ra > 1$ is not comparable to $Ri > \frac{1}{4}$, first of all because Ra and Ri are not the same number, but also because they refer to different basic flows: the metric used to obtain $Ra > 1$ blows up in the limit $\epsilon \to 0$. More importantly, it is only apparent that the condition $Ra > 1$ is free from the ultraviolet problem (because it makes no explicit reference to the shape of the perturbation): the ordering in ϵ indeed assumes a lower bound in the vertical scale of the disturbance.

In summary, *normed stability* (and Arnol'd's second theorem) has been proved for the $N = 1$ case (and the quasi-geostropic models), *formal stability* is here proved for a general layered ageostrophic model, $1\frac{1}{2} \leqslant N < \infty$, and only *normal-mode stability* has been proved in the continuously stratified case, '$N = \infty$'. The interplay of linear/nonlinear stability in the two-dimensional Boussinesq flow is, in my view, still an open problem, which will probably not be resolved with the use of conservation laws, but rather, by numerical integration.

6. The structure of growing perturbations

Laplace tidal equations – whose generalization with the inclusion of buoyancy effects are the *primitive equations* used here – are probably the archetype of ocean dynamical models. When they are linearized in the deviation from a resting ocean, two types of waves are found: Poincaré and Rossby ones (or, rather, gravity waves and vortical modes, respectively, in the absence of Coriolis effects). Two is also the number of conditions that guarantee the stability of the steady solutions of those equations. One may conjecture that growing perturbations from an unstable flow that violate the first condition but satisfy the seond one are Rossby-like. Conversely, an unstable steady solution that fulfills the first one but not the second is presumed to decay through the excitation of Poincaré-like waves.

This conjecture is based in two observations. First, a formally stable state is related to an isolated maximum of pseudo-energy-momentum, $\delta^2 S > 0$, whereas an unstable one has the structure of, say, a saddle point in phase space; growing perturbations represent a route of escape from the basic state along the $\delta^2 S \equiv 0$ subspace. For instance, for a normal mode growing like $e^{\nu t}$, since $\delta^2 S$ is both constant and proportional to $e^{2\nu t}$, then such integral must vanish identically. Second, Poincaré and Rossby perturbations contribute mainly to the wave energy (or wave energy momentum) and wave Casimir parts of $\delta^2 S$, respectively. That is certainly the case for their contribution to either part of the pseudomomentum relative to a resting ocean (Ripa 1982); pseudoenergy relative to a resting ocean is but regular energy.

Consequently, if $\delta^2 E$ is positive definite (the second condition is fulfilled) then it takes a Rossby wave to make $\delta^2 C$ negative enough for a perturbation to grow. Conversely, if $\delta^2 C$ is positive definite (the first condition is satisfied) it takes a Poincaré wave to make $\delta^2 E$ negative enough. For an x-symmetric (axisymmetric) basic state, in the last two sentences $\delta^2 E$ should be replaced by $\delta^2 E - \alpha \delta^2 M$ ($\delta^2 E - \omega \delta^2 A$), where α (ω) is any value such that only one of the conditions is violated.

More precisely, a Poincaré wave is characterized by $\delta\xi \approx 0$ ($\equiv h\Delta q$; see (11)), whereas a Rossby one by $\delta u_x + \delta v_y \approx 0$, by analogy with free waves in a resting ocean (e.g. see Ripa 1981); a quantitative measure of the Poincaré/Rossby dominance would then be the ratio

$$\chi = \frac{\langle (\delta u_x + \delta v_y)^2 \rangle}{\langle \delta \xi^2 \rangle}. \tag{28}$$

The transparency of the wave Casimir to Poincaré disturbances is then obvious. On the other hand, if Rossby waves have a structure similar to that in a quasi-geostropic model (I am working with expressions for continuous stratification for simplicity, but the argument follows the same lines in a layered system): $\delta u \approx -\partial_y \varphi, \delta v \approx \partial_x \varphi$, $\delta h \approx -\partial_z (f_0 \mathcal{N}^{-2} \partial_z \varphi)$, it follows that their contribution $\alpha\, \delta^2 M$, or $\omega\, \delta^2 A$ vanishes, and that the integral of $\vec{U} \cdot \delta\vec{u}\, \delta h$ is also negligible, as long as the horizontal (vertical) shear of the basic flow is much smaller than $|f_0|$ ($\mathcal{N}$).

Some evidence in support of the conjecture follows, albeit not in the form of the ratio χ.

Marinone & Ripa (1984) studied unstable easterly equatorial jets in a $1\frac{1}{2}$-layer model. A narrow jet violated the first condition, ($1a$), but not the second one, ($1b$); the opposite was true for a jet with a width equal to the deformation radius: the structure of growing perturbations in each case was like those just described (Rossby- and Poincaré-like, respectively). This is a good test for the conjecture, because in the equatorial wave guide there is not always a clear cut distinction between Rossby and Poincaré waves (i.e. it is possible to go smoothly from one to the other type), on account of the zero crossing of the Coriolis 'parameter'.

Sakai (1989) studied the stability of the 2-layer model in a channel, for the case in which the basic flow has no horizontal shear. The first condition, ($2a$), is violated for any value of α; fulfilment of the second condition, ($1c$), for the optimum value of α, requires the Froude number Fr to be smaller than $\sqrt{2}$ (i.e. equation (27) for $H_1 = H_2$). Now, for $Fr \ll \sqrt{2}$, Sakai finds only baroclinic instability, for which growing modes are Rossby-like, as conjectured here. Since for $Fr \geqslant \sqrt{2}$ *both* conditions are violated, for any value of α, the conjecture does not apply. However, it is interesting to point out that for $Fr \gtrsim \sqrt{2}$ Sakai finds a mixed mode, Rossby–Kelvin, of instability, and that Kelvin–Helmholtz instability is found for $Fr \gg \sqrt{2}$.

Finally, Barth (1989*a*, *b*) studied the stability of a coastal front using the so-called geostropic momentum equations, which are quite different from the primitive equations used here, in the sense that the main balance is assumed to be geostrophic (the ageostrophic velocity components are diagnostic variables). Nevertheless, Barth finds exactly the same conditions, namely (2*a*) and (1*c*). He then studies an unstable case in which only the first condition is violated (uniform potential vorticity in the upper layer) and finds a 'significant ageostropic component of the velocity field' in the growing perturbations, in spite of the equations used; it would be interesting to compute the value of χ for this solution, and to repeat the calculation with the full primitive equations.

7. Discussion

Stability is such a mighty, but still model- and definition-dependent, concept. Thus, a certain flow might be stable in the framework of a particular model and become unstable once this is changed, say by relaxing one of the assumptions with which it was set up. For instance Kelvin–Helmholtz instability can be prevented in a two-layer hydrostatic model, if the shear does not exceed a certain threshold, (27), but the flow will always be unstable to horizontally short enough perturbations, as soon as the hydrostatic approximation is not made. As another example, for a certain current the choice of whether it will be unstable could depend on whether the quasi-geostropic approximation is invoked.

This dependence on the model, even though it makes the application to observations more difficult, could be beneficial, because models usually differ in the physics they represent. For instance, one way of describing the contrast between a primitive equation and a quasi-geostrophic model is by pointing out that the latter lacks the degrees of freedom of the former corresponding to Poincaré waves. Therefore, if some flow is stable according to a quasi-geostropic model, but not so in a primitive-equation one, we might expect that growing perturbations will in some way resemble Poincaré or gravity waves, as conjectured in §6.

What cannot be done, however, is to use the results of one model in the domain of another model. Self evident as it may seem, this principle is ignored quite often; one sees researchers using $\beta - U_{yy}$ in the equatorial region, or authors astonished to find a front (which cannot be studied by a quasi-geostrophic model) to be unstable even though the potential vorticity is uniform. Problems in which the horizontal divergence is important are better studied by a primitive-equation model. Two stability conditions are then needed. The first one, the equivalent to Rayleigh's inflection point theorem, is shared with the quasi-geostrophic models; this condition assures that a quadratic wave energy momentum is bounded from above. The second condition is present in order to guarantee that wave energy momentum is positive definite, i.e. bounded from below. Thus in the example of the unstable front, or in the Kelvin–Helmholtz problem, growing perturbations have negative or vanishing wave energy and momentum; the sign of the energy transfer between mean flow and wave might not be very enlightening in such a case.

The second stability condition, the one that assures that the wave energy momentum is positive definite, is harder and harder to satisfy as the number of layers is increased. Does that mean that a current with vertical shear is always unstable, if vertical resolution of the model is good enough? Or does this point to a limitation of Arnol'd's method? Probably the second: the concept of stability is not independent of the particular definition used, i.e. of the measure of the system's departure from the basic state. The metric used in this paper is a wave energy momentum, because the stability conditions are derived from the conservation laws. A vertically sheared parallel flow, in the non-rotating case, is stable in the normal-mode sense (the frequency of all infinitesimal waves have non-positive imaginary part) if the Richardson number is everywhere larger than one fourth; yet, it has not been possible to prove this stability with the methods of this paper.

Stability is a model- and definition-dependent concept. In exploring the former with different definitions of stability, we end up, many times, surveying the range of diverse methods, rather than establishing absolute truths on the physics of a given system.

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Appendix. Potential energy

It is useful to spell out the relationship between the density and pressure fields, h_j and p_j, from (4). Write

$$h_j = \tilde{h}_j + \zeta_{j-1} - \zeta_j, \tag{A 1}$$

where the $\tilde{h}_j$ are constants. In a layer that is horizontally unbounded, a typical choice for $\tilde{h}$ is the average thickness (i.e. the average ζ vanishes). On the other hand, for a layer with a finite domain (e.g. bounded by outcropping interfaces), $\tilde{h} = 0$ could be chosen. Strictly speaking, the $\tilde{h}_j$ are arbitrary.

The inverse of (A 1) is

$$\zeta_i \equiv \zeta_n + \sum_{j=i+1}^{n} (h_j - \tilde{h}_j) \quad (i = 0, \ldots, n-1). \tag{A 2}$$

On the other hand, the hydrostatic balance (4*a*) implies

$$p_i \equiv \sum_{j=0}^{i-1} g'_j \zeta_j, \tag{A 3}$$

where it is understood $g'_0 \equiv g$, and ζ_0 is the sea-surface elevation.

Consider first the case of *reduced gravity*: $p_{n+1} = 0$ implies

$$\sum_{j=0}^{n} g'_j \zeta_j \equiv 0. \tag{A 4}$$

Solving this equation for ζ_n, an dusing (A 2) in (A 3), it follows that

$$p_i \equiv \sum_{j=1}^{n} G_{ij}(h_j - \tilde{h}_j) \quad (i = 1, \ldots, n), \tag{A 5}$$

with

$$G_{ij} \equiv G_{ji} \equiv \left(\sum_{k=0}^{a-1} g'_k\right)\left(\sum_{k=b}^{n} g'_k\right)\left(\sum_{k=0}^{n} g'_k\right)^{-1}, \tag{A 6}$$

and where a and b are the minimum and maximum of $\{i,j\}$, respectively. In the rigid-lid approximation, $g \to \infty$, G_{ij} reduces to just the middle summation.

Consider now the *rigid-bottom* case: instead of being determined by (A 4), $\zeta_n(\vec{x})$ is now given by a time-independent function, which represents the topography. Without the rigid-lid approximation, simply using (A 2) in (A 3) results in

$$p_i \equiv G_{in}\,\zeta_n + \sum_{j=1}^{n} G_{ij}(h_j - \tilde{h}_j), \tag{A 7}$$

where now G_{ij} is given by the first summation in (A 6); recall that the first term is time-independent. Finally, *with* the rigid-lid approximation, using

$$\zeta_i \equiv -\sum_{j=1}^{i} (h_j - \tilde{h}_j) \tag{A 8}$$

in (A 3),

$$p_i - p_n = \sum_{j=1}^{n} G_{ij}(h_j - \tilde{h}_j) \tag{A 9}$$

is obtained, with the same G_{ij}.

Equations (A 5), (A 7) or (A 9) are not explicitly used in the main text; the only important point about them is that the matrix G_{ij} is symmetric, from where it follows that

$$\sum_{j=1}^{n}\left[h_j \frac{\partial p_j}{\partial t} - p_j \frac{\partial h_j}{\partial t}\right] = 0 \tag{A 10}$$

up to a term whose horizontal integral is an exact time derivative (i.e. that obtained replacing h_j by $\tilde{h}_j$). Notice that this is true for all four possible combinations of top and bottom boundary conditions: with or without a rigid lid and for reduced gravity or a rigid bottom. In the particular case of a rigid lid *and* rigid bottom, p_j must be replaced by $p_j - p_n$, as in table 1.

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